THEORY OF THE REFLECTION OF THE LIGHT FROM A POINT SOURCE BY A FINITELY CONDUCTING FLAT MIRROR, WITH AN APPLICATION TO RADIOTELEGRAPHY

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Abstract

The investigation starts from the rather complicated integral expressions given by Sommerfeld, Weyl and Bateman for the Hertzian functions (1) and (2) for the field at $B(z_b, \rho_0, o)$ in the first medium, due to an electric or magnetic dipole placed also in the first medium at $A(z_a, o, o)$ above an infinitely extending but finitely conducting mirror (or earth) (second medium). Instead of obtaining approximations for these expressions or developing them in series, (as has been done by several investigators) which still lead to very complicated mathematical expressions without much physical transparency, the writer obtains a solution without any approximations whatsoever in the form of a space integral (15) and (17) having exactly the same form for the magnetic and electric case. The integration extends over the part of space occupied by the second medium below the geometrical image. It is shown that the field in the first medium, apart from the direct radiation from the point source, can be described as due to secondary waves originating in the integration space mentioned, the amplitude of these secondary waves being determined by the amplitude of a primary wave which can be considered to spread from the geometrical image of the point source with a propagation constant and absorption belonging to the second medium. The higher the conductivity of this second medium, the more the primary wave is therefore concentrated near the image of the point source until, for infinite conductivity, it is wholly concentrated at the image itself. Thus a mathematical approximation given by Sommerfeld and Weyl is extended and it is given a physical basis. From the theory developed a certain analogy between 1°. a perfectly smooth but finitely conducting mirror and 2°. an infinitely conducting but not perfectly smooth mirror is difficult to be denied.

§ 1. The problem under consideration consists of the evaluation of the electromagnetic field due to a point source, situated at a certain

height above a horizontal finitely conducting flat mirror. The application to radiotelegraphy concerns an electric or magnetic vertical dipole antenna placed at a certain height above the finitely conducting ground and the field both on the ground and in the space above is investigated. The earth is further assumed to be flat, so that in the wireless case our results are only valid for those distances and wavelengths where the curvature of the earth can be neglected.

This problem has been investigated in two classical papers, one by A. Sommerfeld [1]) in 1909 and one by H. Weyl [2]) in 1919. A very concise exposition is given by Sommerfeld in Frank-von Mises, „Die Differentialgleichungen der Physik" [3]).

Sommerfeld considers the dipole to be placed on the earth and finds a solution in the form of a rather intricate integral. Weyl considers the radiation from the point source as the superposition of an infinity of plane waves from different directions (including imaginary ones). Each of these plane waves is reflected by the earth and the superposition of all the direct and all the reflected waves gives, again in the form of a similar integral, the mathematical expression for the field [4]).

The results of Sommerfeld and Weyl were lateron the subject of many investigations by different writers. Their results were, as a rule, not very transparent on account of the fact that the approximations or developments used were more of a mathematical than physical character.

It is the purpose of the present paper to show that the problem, *without any approximation whatsoever*, is capable of being developed in a different way leading to a solution in the form of a simple space integral which allows a direct physical interpretation.

In originally deriving our results extensive use was made of the heuristic value of the operational or symbolic calculus, but the presentation here will be given with the aid of ordinary analysis.

§ 2. Let the light source or antenna (we will use the terminology of both physical problems arbitrarily) be situated on the z-axis at A

1) A. Sommerfeld, Ann. Physik, **28**, 287, 1909.
See also: Ann. Physik, **81**, 1138, 1926.
2) H. Weyl, Ann. Physik, **60**, 481, 1919.
3) Bd. II (2. Aufl.) 1935, p. 918.
4) For another derivation of Weyl's result see: K. F. Niessen, Ann. Physik, **18**, 893, 1933.

(fig. 1) at a height z_a above the finitely conducting but infinitely thick mirror or earth. We introduce cylindrical coordinates (z, ρ, φ) so that the point source is situated at (z_a, o, o). The field is to be investigated at an arbitrary point B (with coordinates z_b, ρ_0 and o) in the first medium. The indices are here introduced because lateron we shall have to integrate over space and we reserve z, ρ and φ (without index) for that purpose. In the upper or first medium the field can be derived from a Hertzian vector Π_1 satisfying

$$(\Delta + k_1^2)\, \Pi_1 = 0,$$

and, similarly, in the lower or second medium from Π_2 satisfying

$$(\Delta + k_2^2)\, \Pi_2 = 0.$$

As shown by Elias [1]) and Sommerfeld the boundary conditions are different for a magnetic and an electric dipole as

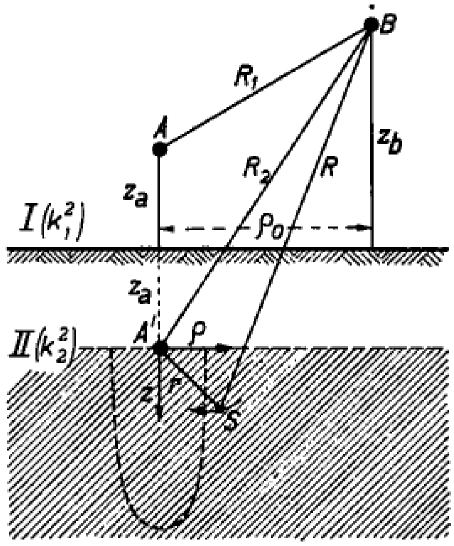

Fig. 1.

radiation source. We will, however, study both cases simultaneously. Limiting ourselves to a consideration of the field in the first medium only and calling the total vector in the magnetic case $\Pi_m = \Pi_{m\,pr} + \Pi_{m\,sec}$, and in the electric case $\Pi_e = \Pi_{e\,pr} + \Pi_{e\,sec}$, following Sommerfeld [2]), we split up each of these vectors in two parts:

1) G. J. Elias, Physica **2**, 207, 361, 1922.

2) See also H. Bateman, „Electrical and optical wavemotion" (Cambridge 1915) page 74, 75.

1) the primary parts: $\Pi_{e\,pr} = e^{jk_1R_1}/R_1$ and $\Pi_{m\,pr} = e^{jk_1R_1}/R_1$,
2) the secondary parts: $\Pi_{e\,sec}$ and $\Pi_{m\,sec}$ due to the presence of the mirror (earth).

The secondary parts are for the electric and magnetic cases given resp. by

$$\Pi_{e\,sec} = \int\limits_{\lambda=0}^{\lambda=\infty} \frac{k_2^2 l - k_1^2 m}{k_2^2 l + k_1^2 m} \cdot e^{-(z_a+z_b)l} \cdot J_0(\lambda\rho_0)\,\frac{\lambda d\lambda}{l} \tag{1}$$

$$\Pi_{m\,sec} = \int\limits_{\lambda=0}^{\lambda=\infty} \frac{l-m}{l+m} \cdot e^{-(z_a+z_b)l} \cdot J_0(\lambda\rho_0)\,\frac{\lambda d\lambda}{l} \tag{2}$$

where (fig. 1):

$$\begin{aligned} R_1^2 &= \rho_0^2 + (z_a - z_b)^2, \\ k_{1,2}^2 &= \frac{1}{c^2}\,(\varepsilon_{1,2}\,\omega^2 + j\omega\sigma_{1,2}), \end{aligned} \tag{3}$$

$\varepsilon_{1,2}$ and $\sigma_{1,2}$ representing the dielectric constant and conductivity of the media I and II resp. while ω is the angular frequency of the waves. It is to be remarked at the outset that both k_1^2 and k_2^2 are lying in the *first* quadrant of the complex plane, whereas the roots $l = \sqrt{\lambda^2 - k_1^2}$ and $m = \sqrt{\lambda^2 - k_2^2}$, as occurring in (1) and (2), are defined to have their *real part positive* for all values of the integration variable $o \leqslant \lambda \leqslant \infty$. The derivation of (1) and (2) by Sommerfeld is based on the following representation of the point source free in space

$$\frac{e^{jk_1\sqrt{\rho_0^2+(z_a-z_b)^2}}}{\sqrt{\rho_0^2+(z_a-z_b)^2}} = \int\limits_{\lambda=0}^{\lambda=\infty} e^{-(z_b-z_a)l} \cdot J_0(\lambda\rho_0) \cdot \frac{\lambda d\lambda}{l}, \qquad z_b - z_a > 0, \tag{4}$$

which integral we will encounter again in what follows. Considering (1) and (2) we observe that Π_e and Π_m satisfy the *reciprocity condition*, i.e. they are not changed when we interchange the position of transmitter and receiver. Moreover the secondary fields as given by $\Pi_{e\,sec}$ and $\Pi_{m\,sec}$ are a function of the combination $z_a + z_b$ only, i.e. the secondary field is not altered when e.g. we lower the height of the transmitter over a certain distance, provided that at the same time we increase the height of the receiver by a similar amount. This fact is obvious for an infinitely conducting mirror or earth, but perhaps it is not for the case under consideration.

As, further, the electrical case Π_e is more complicated than the magnetic case Π_m (as given by (1) and (2)) we will fully work out the electric case while simply indicating the simplifications occurring in the magnetic case.

§ 3. Instead of trying to evaluate the secondary field of the electric case $\Pi_{e\,sec}$, as given by (1), by approximation or by developing it in a series, we will, in order to obtain a simple result of a physical character, proceed along a different road and open up the integral. To this end we remark that, with the above definitions of k_1^2, k_2^2, l and m, the denominator $k_2^2 l + k_1^2 m$ has for all values $o \leqslant \lambda \leqslant \infty$ its *real part positive*. Hence we can write it as a convergent integral as follows:

$$\frac{1}{k_2^2 l + k_1^2 m} = \int_0^\infty e^{-(k^2_2 l + k^2_1 m)\zeta}\, d\zeta,$$

and similarly:

$$\frac{1}{l}\,\frac{k_2^2 l - k_1^2 m}{k_2^2 l + k_1^2 m} = \left(k_2^2 - k_1^2 \frac{m}{l}\right) \int_0^\infty e^{-(k^2_2 l + k^2_1 m)\zeta}\, d\zeta =$$

$$= \frac{1}{k_1^2} \int_0^\infty \left[\frac{\partial}{\partial\zeta}\left(\frac{e^{-k^2_2 l\zeta}}{l}\right) \cdot \frac{\partial}{\partial\zeta}\left(\frac{e^{-k^2_1 m\zeta}}{m}\right) - \left(\frac{e^{-k^2_2 l\zeta}}{l}\right) \cdot \frac{\partial^2}{\partial\zeta^2}\left(\frac{e^{-k^2_1 m\zeta}}{m}\right)\right] d\zeta. \quad (5)$$

Substituting in (1) we obtain:

$$\Pi_{e\,sec} =$$

$$\frac{1}{k_1^2} \int_{\zeta=0}^{\zeta=\infty} \int_{\lambda=0}^{\lambda=\infty} \left[\frac{\partial}{\partial\zeta}\left(\frac{e^{-(k^2_2\zeta + z_a + z_b)l}}{l}\right) \cdot \frac{\partial}{\partial\zeta}\left(\frac{e^{-k^2_1 m\zeta}}{m}\right) - \right.$$

$$\left. - \left(\frac{e^{-(k^2_2\zeta + z_a + z_b)l}}{l}\right) \cdot \frac{\partial^2}{\partial\zeta^2}\left(\frac{e^{-k^2_1 m\zeta}}{m}\right)\right] J_0(\lambda\rho_0)\,.\,\lambda d\lambda d\zeta. \quad (6)$$

The next step is a transformation of the expression

$$\frac{e^{-k^2_1 m\zeta}}{m} = \frac{e^{-k^2_1\zeta\sqrt{\lambda^2 - k^2_2}}}{\sqrt{\lambda^2 - k_2^2}}$$

occurring twice in (6).

To this end we make use of a relation, similar to (4):

$$\frac{e^{-f\sqrt{\lambda^2-k_2^2}}}{\sqrt{\lambda^2-k_2^2}} = \int_0^\infty \frac{e^{jk_2\sqrt{s^2+f^2}}}{\sqrt{s^2+f^2}} \cdot J_0(\lambda s) \, . \, sds \tag{7}$$

which was proved by L a m b [1]) for real positive f, and which follows from H a n k e l's inversion formula for integrals over Besselfunctions [2]).

A comparison shows that in our case $f = k_1^2\zeta$ is complex, but its real part, according to the definition of k_1^2 is positive [3]). We observe that in our case $e^{-k_1^2\zeta\sqrt{\lambda^2-k_2^2}}/\sqrt{\lambda^2-k_2^2}$ therefore tends uniformly and exponentially to zero for $\lambda \to \infty$ and for $\zeta \to \infty$, because, according to the definition, $Re\ \sqrt{\lambda^2-k_2^2} > 0$ and thus $\sqrt{\lambda^2-k_2^2}$ lies in the fourth quadrant. We therefore apply (7) also for complex $f = k_1^2\,\zeta$ and write

$$\frac{e^{-k_1^2\zeta m}}{m} = \frac{e^{-k_1^2\zeta\sqrt{\lambda^2-k_2^2}}}{\sqrt{\lambda^2-k_2^2}} = \int_{s=0}^{s=\infty} \frac{e^{jk_2\sqrt{s^2+(k_1^2\zeta)^2}}}{\sqrt{s^2+(k_1^2\zeta)^2}} \cdot J_0\,(\lambda s). \, sds, \tag{8}$$

provided we again define $\sqrt{s^2+(k_1^2\zeta)^2}$ to have its real part positive, so that $e^{jk_2\sqrt{s^2+(k_1^2\zeta)^2}}$ tends uniformly and exponentially to zero both for $s \to \infty$ *and* $\zeta \to \infty$, k_2 lying in the first quadrant.

Hence we can write (6) as

$$\Pi_{e\,sec} = \frac{1}{k_1^2} \int_{\zeta=0}^{\zeta=\infty} \int_{\lambda=0}^{\lambda=\infty} \int_{s=0}^{s=\infty} \left[\frac{\partial}{\partial\zeta}\left(\frac{e^{-(k_1^2\zeta+z_a+z_b)l}}{l}\right) \cdot \frac{\partial}{\partial\zeta}\left(\frac{e^{jk_2\sqrt{s^2+(k_1^2\zeta)^2}}}{\sqrt{s^2+(k_1^2\zeta)^2}}\right) - \left(\frac{e^{-(k_1^2\zeta+z_a+z_b)l}}{l}\right) \cdot \frac{\partial^2}{\partial\zeta^2}\left(\frac{e^{jk_2\sqrt{s^2+(k_1^2\zeta)^2}}}{\sqrt{s^2+(k_1^2\zeta)^2}}\right)\right] \cdot J_0(\lambda\rho_0) \cdot J_0(\lambda s) \, . \, d\zeta\, \lambda d\lambda\, sds. \tag{9}$$

Further, substituting in (9) for the product of the Besselfunction the addition theorem

$$J_0(\lambda\rho_0) \cdot J_0(\lambda s) = \frac{1}{2\pi}\int_0^{2\pi} J_0(\lambda\sqrt{\rho_0^2 - 2s\rho_0\cos\varphi + s^2})\, d\varphi,$$

1) H. L a m b, Proc. London math. Soc. (2) 7, 140 (1909).

2) See e.g. H. B a t e m a n, „Partial differential equations" (Cambr. 1932) p. 411.

3) G. N. W a t s o n in his book „*Besselfunctions*" remarks (page 416) that „with certain limitations" f may a.o. be complex.

and remembering that $l = \sqrt{\lambda^2 - k_1^2}$ it is possible, with the aid of (4) now to effect in (9) the integration with respect to λ, because we have

$$\int\limits_{\lambda=0}^{\lambda=\infty} \frac{e^{-(k_2^2\zeta + z_a + z_b)\sqrt{\lambda^2 - k_1^2}}}{\sqrt{\lambda^2 - k_1^2}} \cdot J_0(\lambda \sqrt{\rho_0^2 - 2s\,\rho_0 \cos\varphi + s^2}\,)\,\lambda d\lambda = \frac{e^{jk_1R'}}{R'},$$

where

$$R' = \sqrt{\rho_0^2 - 2s\,\rho_0 \cos\varphi + s^2 + (z_a + z_b + k_2^2\zeta)^2},$$

this root being again defined to have its real part positive.

Writing further, for obvious reasons, for the integration variable ρ instead of s, we thus obtain from (9)

$$\Pi_{e\,sec} = \frac{1}{2\pi\,k_1^2} \int\limits_{\zeta=0}^{\zeta=\infty} \int\limits_{\rho=0}^{\rho=\infty} \int\limits_{\phi=0}^{\phi=2\pi} \left[\frac{\partial}{\partial\zeta}\left(\frac{e^{jk_1R'}}{R'}\right) \cdot \frac{\partial}{\partial\zeta}\left(\frac{e^{jk_2r'}}{r'}\right) - \right.$$
$$\left. - \left(\frac{e^{jk_1R'}}{R'}\right) \cdot \frac{\partial^2}{\partial\zeta^2}\left(\frac{e^{jk_2r'}}{r'}\right)\right] \rho\, d\rho\, d\zeta\, d\varphi \qquad (10)$$

where we defined

$$r' = \sqrt{\rho^2 + (k_1^2\zeta)^2}, \qquad (10a)$$

$$R' = \sqrt{\rho_0^2 - 2\rho\rho_0 \cos\varphi + \rho^2 + (z_a + z_b + k_2^2\zeta)^2}, \qquad (10b)$$

both roots having their real part positive.

Before proceeding further with (10) (which represents the *electric case*,) we first remark that, if we had treated the *magnetic case* (2) in exactly the same way, we would have obtained the quite similar expression:

$$\Pi_{m\,sec} = \frac{1}{2\pi} \int\limits_{z=0}^{z=\infty} \int\limits_{\rho=0}^{\rho=\infty} \int\limits_{\phi=0}^{\phi=2\pi} \left[\frac{\partial}{\partial z}\left(\frac{e^{jk_1R}}{R}\right) \cdot \frac{\partial}{\partial z}\left(\frac{e^{jk_2r}}{r}\right) - \right.$$
$$\left. - \left(\frac{e^{jk_1R}}{R}\right) \cdot \frac{\partial^2}{\partial z^2}\left(\frac{e^{jk_2r}}{r}\right)\right] \rho\, d\rho\, dz\, d\varphi \qquad (11)$$

but, this time with the *real* positive distances:

$$\left.\begin{aligned} r &= \sqrt{\rho^2 + z^2}, \\ R &= \sqrt{\rho_0^2 - 2\rho\rho_0 \cos\varphi + \rho^2 + (z_a + z_b + z)^2}. \end{aligned}\right\} \qquad (12)$$

The nomenclature of the integration variables ρ, z, φ in the integral

(11) has obviously been chosen such as to clearly express the fact that (11) represents a *space-integral.* Hence writing therefore $\rho d\rho\, dz\, d\varphi = d\tau$, the complete and exact solution for the *magnetic case* is:

$$\Pi_m = \frac{e^{jk_1R_1}}{R_1} + \frac{1}{2\pi}\int\left[\frac{\partial}{\partial z}\left(\frac{e^{jk_1R}}{R}\right)\cdot\frac{\partial}{\partial z}\left(\frac{e^{jk_2r}}{r}\right) - \left(\frac{e^{jk_1R}}{R}\right)\cdot\frac{\partial^2}{\partial z^2}\left(\frac{e^{jk_2r}}{r}\right)\right]d\tau, \quad (13)$$

with the real distances R and r as defined by (12). As the integration variable z ($o \leqslant z \leqslant \infty$) occurs in (11) a.o. in the combination $z_a + z_b + z$, it is obvious (see fig. 1) that we obtain the simplest geometrical interpretation if we define the total integration to be extended over the *half space below the geometric image A' of A*, where, as usual, A' lies the same distance z_a *below* the surface of the mirror (or earth) as the point source (transmitting antenna) A lies above it, and z is measured *positive downwards* from the image A'.

This geometrical interpretation of (13) is represented in fig. 1. To summarize we have: A is the point source, B is the point where we wish to determine the field. Construct the image A' of A. Consider an arbitrary point S (in general not in the plane of the paper) in the second medium but *below* the level of A'. The distance $A'S$ is the r, and the distance BS is the R of (13). The integration is to be extended over the (shaded) half space below the geometrical image A'.

(13) can still further be simplified. Introducing the distance R_2 (fig. 1) between the receiver B and the geometrical image A' of the point source A we can, with (4), write

$$\frac{e^{jk_1R_2}}{R_2} = \int_{\lambda=0}^{\lambda=\infty} e^{-(z_a+z_b)l}\cdot J_0(\lambda\rho_0)\,\frac{\lambda d\lambda}{l}\,. \qquad z_a+z_b>0 \quad (14)$$

We do not change its value if we introduce in this integral the factor $(l+m)/(l+m)$ and write this again in the form like (5). Proceeding then as above, we obtain

$$\frac{e^{jk_1R_2}}{R_2} = \frac{1}{2\pi}\oint\left[\frac{\partial}{\partial z}\left(\frac{e^{jk_1R}}{R}\right)\cdot\frac{\partial}{\partial z}\left(\frac{e^{jk_2r}}{r}\right) + \left(\frac{e^{jk_1R}}{R}\right)\cdot\frac{\partial^2}{\partial z^2}\left(\frac{e^{jk_2r}}{r}\right)\right]d\tau\,.. \quad (15)$$

which has the same form as (13) only with the minus sign replaced by a plus sign. *Combining* (13) *and* (15) *we thus obtain for the magnetic case our final exact result (which contains no approximations whatsoever) in the two forms:*

$$\Pi_m = \frac{e^{jk_1R_1}}{R_1} - \frac{e^{jk_1R_2}}{R_2} + \frac{1}{\pi}\int \frac{\partial}{\partial z}\left(\frac{e^{jk_1R}}{R}\right)\cdot\frac{\partial}{\partial z}\left(\frac{e^{jk_2r}}{r}\right).\,d\tau \qquad (15a)$$

$$= \frac{e^{jk_1R_1}}{R_1} + \frac{e^{jk_1R_2}}{R_2} - \frac{1}{\pi}\int \frac{e^{jk_1R}}{R}\cdot\frac{\partial^2}{\partial z^2}\left(\frac{e^{jk_2r}}{r}\right).\,d\tau \qquad (15b)$$

Especially the last expression (15*b*), allows of a very simple physical interpretation; the first two terms $e^{jk_1R_1}/R_1 + e^{jk_1R_2}/R_2$ represent what would be obtained in the case of an infinitely conducting mirror or earth. In addition to these terms we have a third term: the (space) integral. This clearly represents a wave $\partial^2/\partial z^2(e^{jk_2r}/r)$ spreading from the geometrical image A' of the point source A with a velocity and absorption (k_2) belonging to the *second* medium, while each of the points of this second medium *below the geometrical image* A' can, according to the factor e^{jk_1R}/R, be considered to send secondary waves with a propagation constant k_1 belonging to the first medium, to the observer (or receiver) at B.

§ 4. Having thus solved the magnetic case Π_m we now return to the electric case Π_e and consider again (10) with the definitions (10*a*) and (10*b*). It is seen that the electric case is formally completely aequivalent to the magnetic case, only in the electric case we encounter the *complex* distances R' and r' (with real part positive) as given by (10*a*) and (10*b*) instead of the positive *real* distances R and r (12) of the magnetic case. The occurrence of complex distances however is well known from the theory of anisotropic absorbing crystals and represents as such nothing peculiar. Further, a similar artifice as used to obtain (15), by inserting in the integral (14) the factor $(k_2^2l + k_1^2m)/(k_2^2l + k_1^2m)$ and treating it as before, leads to

$$\frac{e^{jk_1R_2}}{R_2} = \frac{1}{2\pi k_1^2}\int\limits_{\zeta=0}^{\phi=\infty}\int\limits_{\rho=0}^{\rho=\infty}\int\limits_{\phi=0}^{\phi=2\pi}\left[\frac{\partial}{\partial\zeta}\left(\frac{e^{jk_1R'}}{R'}\right)\cdot\frac{\partial}{\partial\zeta}\left(\frac{e^{jk_2r'}}{r'}\right) + \right.$$
$$\left. + \left(\frac{e^{jk_1R'}}{R'}\right)\cdot\frac{\partial^2}{\partial\zeta^2}\left(\frac{e^{jk_2r'}}{r'}\right)\right]\rho d\rho\, d\zeta\, d\varphi. \qquad (16)$$

If moreover, as is actually the case in our optical or radio problem, k_1^2 is real (non-absorbing first medium) we can in (10), (10*a*), (10*b*) and (16) replace $k_1^2\zeta$ by z, so that the integration is again extended over the formerly indicated *real* half space with space element $d\tau$.

Introducing further the refractory index n of the second medium, given by

$$n^2 = k_2^2/k_1^2,$$

we obtain as *exact solution for the electric case instead of* (15*a*) *and* (15*b*) *the similar space integrals* (*again without any approximation*)

$$\boxed{\begin{aligned}\Pi_e &= \frac{e^{jk_1R_1}}{R_1} - \frac{e^{jk_1R_2}}{R_2} + \frac{1}{\pi}\int \frac{\partial}{\partial z}\left(\frac{e^{jkR_i}}{R_i}\right)\cdot\frac{\partial}{\partial z}\left(\frac{e^{jk_2r}}{r}\right)d\tau, \qquad (17a)\\ &= \frac{e^{jk_1R_1}}{R_1} + \frac{e^{jk_1R_2}}{R_2} - \frac{1}{\pi}\int \frac{e^{jk_1R_i}}{R_i}\cdot\frac{\partial^2}{\partial z^2}\left(\frac{e^{jk_2r}}{r}\right)d\tau, \qquad (17b)\end{aligned}}$$

where, as before

$$r = \sqrt{\rho^2 + z^2}, \tag{18a}$$

but now with the complex distance R_i (*with positive real part*) *given by*

$$R_i = \sqrt{\rho_0^2 - 2\rho\rho_0\cos\varphi + \rho^2 + (z_a + z_b + n^2 z)^2}. \tag{18b}$$

§ 5. The electric case being of greater practical importance than the magnetic case, it is of interest to investigate, how the expressions (17) for Π_e can still further be simplified when we assume the mirror (or earth) to be highly, but not infinitely, conducting, i.e. k_2^2 having a great imaginary part. Then, due to the high absorption, the wave $\partial^2/\partial z^2.(e^{jk_2r}/r)$ in (17*b*) will be rapidly attenuated, so that it only contributes to the total value of the integral for small values of r, i.e. only *that* part of the second medium in the immediate neighbourhood (but below the level) of the geometrical image of the point source can then be considered to send secondary waves to the observer. This reasoning is in complete accord with the usual view where, in the limit when the conductivity of the mirror (or earth) becomes infinite, the secondary waves apparently originate exactly at the geometrical image *only*.

For high, but not infinite, conductivity it is therefore appropriate, unless k_1R_2 is small, to neglect in the expression (18*b*) for R_i the terms containing ρ. Then in (17*b*) the integration with respect to φ can at once be effected, yielding

$$\Pi_e = \frac{e^{jk_1R_1}}{R_1} + \frac{e^{jk_1R_2}}{R_2} -$$

$$-2\int_{\rho=0}^{\rho=\infty}\int_{z=0}^{z=\infty} \frac{e^{jk_1\sqrt{\rho_0^2+(z_a+z_b+n^2z)^2}}}{\sqrt{\rho_0^2+(z_a+z_b+n^2z)^2}}\cdot\frac{\partial^2}{\partial z^2}\left(\frac{e^{jk_2r}}{r}\right).\,\rho d\rho\, dz. \tag{19}$$

Further, also the integration with respect to ρ can be performed, giving

$$\Pi_e = \frac{e^{jk_1R_1}}{R_1} + \frac{e^{jk_1R_2}}{R_2} + 2\int_{z=0}^{z=\infty} \frac{e^{jk_1\sqrt{\rho_0^2+(z_a+z_b+n^2z)^2}}}{\sqrt{\rho_0^2+(z_a+z_b+n^2z)^2}} \cdot \frac{\partial}{\partial z}(e^{jk_2z}) \,.\, dz, \quad (20)$$

or, writing

$$z_a + z_b + n^2 z = \zeta,$$
$$jk_1^2/k_2 = a,$$

(20) becomes

$$\Pi_e = \frac{e^{jk_1R_1}}{R_1} + \frac{e^{jk_1R_2}}{R_2} + 2a\, e^{-a(z_a+z_b)} \int_{\zeta=z_a+z_b}^{\zeta=z_a+z_b+n^2\infty} \frac{e^{jk_1\sqrt{\rho_0^2+\zeta^2}}}{\sqrt{\rho_0^2+\zeta^2}} \,.\, e^{a\zeta}\, d\zeta. \quad (21)$$

The derivation of this expression extends in a physical way a purely mathematical approximation obtained by Sommerfeld and Weyl [1]).

Finally attention may be drawn to the integral in (17*b*), which, as explained before, may be interpreted physically as a wave $\partial^2/\partial z^2.(e^{jk_2r}/r)$ spreading from the geometrical image A' of the point source A (fig. 1) while all points of the second medium below the level of this geometrical image, apparently send secondary waves to the observer at B. However, when the second medium has a high refractive index, i.e. when $|n^2| \gg 1$, the distance R_i from an arbitrary point S in the second medium to the observer, as given by (18*b*) contains z multiplied with n^2, i.e. to the observer the vertical part of the distance R_i below the level of A' counts multiplied with n^2. The wave originating from A' will therefore be observed at B as elongated vertically, somewhat as shown by the dotted ellips in fig. 1. Remembering e.g. the vertically very elongated image of the moon over a wind rippled lake, a certain analogy between: 1°. a finitely conducting but perfectly smooth mirror and, 2°. a mirror of infinite conductivity but not being perfectly smooth, is difficult to be denied

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1) Frank-Von Mises, II, page 936, 937, formula (40).